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# The Optimal $L_1$ Problem for Generalized Polynomial Monosplines and a Related Problem

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## 1. INTRODUCTION

In this paper we investigate generalized polynomial monosplines with fixed multiple knots and free multiple zeros which have minimal $L_1$-norm. We call this subject the Optimal $L_1$ Problem for Generalized Polynomial Monosplines. We prove that there exists a unique monospline of this type. Related to this problem is a problem for monosplines with free knots which have a set of prescribed zeros. Also in this case the existence of a unique solution is shown. In a future paper, we expect to utilize these results to solve the Extended Optimal $L_1$ Problem for Generalized Polynomial Monosplines, i.e., with both free knots and zeros.

Some of the early work in this area was accomplished by Karlin and Pinkus [8, 9], Jetter and Lange [6, 7], and Strauss [12, 13]. For Extended Cheybeshev Systems the Extended Optimal $L_1$ Problem was solved by Bojanov, *et al.* [4]. Strauss [13a] dealt with $L_1$ approximation for polynomial monosplines with fixed knots of multiplicity two. Michelli [10a] and Barrar and Loeb [1a] investigated $L_1$ approximation for weak Chebyshev systems.

Related to the Optimal $L_1$ Problem for Generalized Polynomial Monosplines is a problem which can either be thought of as a generalization of the Fundamental Theorem of Algebra for monosplines or as a Generalized Gaussian Quadrature Formula. The close relationship between these problems, in the case of an Extended Chebyshev System was emphasized by Braess [5, Chapter I, Sect. 4; Chapter VIII, Sect. 4].

We use a variational approach to show the existence of a solution to the optimal $L_1$ problem (Theorem 1). We then use the relationship between the problems to demonstrate the uniqueness of the solution (Theorem 3).

Schoenberg was the first to state the Fundamental Theorem of Algebra



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for Monosplines. Karlin and Schumaker proved the existence and uniqueness of monosplines with simple knots and prescribed multiple zeros. Micchelli solved the existence and uniqueness problem for multiple knots and simple zeros. (See Michelli [10] for references). Barrar and Loeb [1], proved the uniqueness theorem for the case of multiple zeros and knots of odd multiplicity. They also established the existence in more general cases than those treated by Karlin and Schumaker or Micchelli. Recently Zhensykbaev [14] by using the Brouwer Fixed Point Theorem has given a necessary and sufficient condition for the existence of a solution to the problem with multiple zeros and knots of odd multiplicity.

In this paper we establish in the general case of multiple zeros and arbitrary multiplicities of the knots necessary and sufficient conditions that a solution exists to the Fundamental Theorem of Algebra problem and prove that at most one solution exists.

We will discuss two problems.

## Type I Problem

The problem is to determine among the set of monosplines with given fixed knots which have a certain number of (free) zeros one with minimal $L_1$-norm.

Find zeros $x_i$, $i=1, ..., r+1$,

$$0=x_0<x_1<x_2< \cdots <x_r<x_{r+1}=1$$

with given multiplicities $m_i$, when we are given knots $v_i$, $i=0, 1, ..., s$,

$$0=v_0<v_1<v_2< \cdots <v_s<v_{s+1}=1$$

with multiplicities $n_i$ such that if

$$G(x)=\sum_{i=0}^{s} (-1)^{g_i} \int_{v_i}^{v_{i+1}} \Phi_m(x, v)\, dv,$$

$$g_0=0, \quad g_i=g_{i-1}+(n_i+\sigma_i), \quad i=1, ..., s,$$

with $\sigma_i=0$ or $+1$ (to be specified later, see, e.g., Eq. (18)) then

$$F(x^*, x)=G(x)+\sum_{i=0}^{s}\sum_{j=0}^{n_i-1} b_{ij}(x^*)\, \Phi_m^j(x, v_i), \qquad x^*=(x_1, x_2, ..., x_r) \quad (1)$$

has zeros of order $m_i$ at the $x_i$, and $\int_0^1 |F(x^*, x)|\, dx$ is minimum.

Related to this is

## TYPE II PROBLEM

The problem is to determine a unique monospline with given zeros.

Find distinct knots $t_i$, $i=0, ..., r$, with multiplicity $\bar{n}_i$

$$0=t_0<t_1<t_2<\cdots<t_r<t_{r+1}=1$$

(and corresponding constants $a_{ij}$) such that

$$M(y)=\bar{G}(y)+\sum_{i=0}^{r}\sum_{j=0}^{\bar{n}_i-1} a_{ij}\Phi_m^j(y, t_i) \tag{2}$$[1]

has zeros of order $\bar{m}_i$ at given $y_i$, $i=1, ..., s+1$, $\bar{m}_0=m-\bar{n}_0$

$$0=y_0<y_1\cdots<y_s<y_{s+1}=1$$

with

$$\bar{G}(y)=\sum_{i=0}^{r}(-1)^{\bar{g}_i}\int_{t_i}^{t_{i+1}}\Phi_m(y, t)\,dt,$$

$$\bar{g}_0=0, \quad \bar{g}_i=\bar{g}_{i-1}+(\bar{n}_i+1), \quad i=1, ..., r,$$

i.e., here $\sigma_i=1$, $i=1, ..., r$.

We have used the following notation in the statement of these two types of problems:

$$\Phi_m(x, v)=\frac{(x-v)_+^{m-1}}{(m-1)!}, \qquad \Phi_m^j(x, v)=\frac{\partial^j}{\partial v^j}\Phi_m(x, v).$$

As emphasized by Braess in his book [5], in the context of Extended Chebyshev Systems, these problems are closely related.

We now show that these problems can be broken up into simpler problems, which we call indecomposable problems. We give the precise definitions below. We begin with some general remarks applicable to both types of problems.

If $F(x)$ is piecewise, between the knots $v_i$ of multiplicity $n_i$, an $m$th degree polynomial with leading coefficient $\pm A$, $A>0$, we define for the purpose of the statement of the Budan–Fourier Theorem

$$\sigma_i=\begin{cases}1 & \text{if } (-1)^{n_i+1}\operatorname{sgn} D_-^m F(v_i)=\operatorname{sgn} D_+^m F(v_i)\neq 0\\ 0 & \text{if } (-1)^{n_i}\operatorname{sgn} D_-^m F(v_i)=\operatorname{sgn} D_+^m F(v_i)\neq 0.\end{cases}$$

We call such an $F(x)$ a generalized monospline. Thus both the $F(x^*, x)$ of

[1] An important special case is when $\bar{n}_i=0$. Then $a_{ij}=0$ for all $j$.

problems of Type I and the $M(y)$ of problems of Type II are special cases. We will use the same notation for the knots and zeros of $F(x)$ as that used for $F(x^*, x)$ in problems of Type I. From arguments used in [10, Prop. 1] and [11, Theorem 8.43], we have

LEMMA 1. (Budan–Fourier Theorem). *$Z_{(a,b)}(F) \leqslant m + \sum (n_i + \sigma_i) - S^+ LF(a) - S^+ RF(b)$ where the sum is over all knots contained in the open interval $(a, b)$. The following notation is used:*

*$Z_{(a,b)}(F)$ is the total number of zeros of $F$ in the interval $(a, b)$, using the zero convention of* [11, *Definition* 8.42].

*Also,*

$$RF(a) = [F(a+), -D_+ F(a), ..., (-1)^m D_+^m F(a)]$$

$$LF(b) = [F(b-), D_- F(b), ..., D_-^m F(b)].$$

*Thus we say $F(x)$ of Lemma* 1 *has a full set of zeros if*

$$\sum_{j=0}^{r+1} m_j = m + \sum_{i=1}^{s} (n_i + \sigma_i) = N_1. \tag{3}$$

We also have the following Corollary of Lemma 1 (see Micchelli [10, Cor. 2] and Schumaker [11, Theorem 8.44]).

COROLLARY 1. *If $F(x)$ of Lemma* 1 *has a full set of zeros, with the zeros arranged as $0 = \bar{x}_1 \leqslant \bar{x}_2 \leqslant \cdots \leqslant \bar{x}_{N_1} = 1$, then for each knot $v_i$*

$$\bar{x}_{l_i} \leqslant v_i \leqslant \bar{x}_{m + l_{i-1} + 1}, \qquad i = 1, ..., s, \tag{4}$$

*where $l_i = \sum_{j=1}^{i} (n_j + \sigma_j)$, $l_0 = 0$. Both sides of* (4) *are inequalities if $n_i < m$, and if a zero occurs at $v_i$ it is at most of order $m - n_i$. Moreover if $n_i = m$ and $\sigma_i = 1$, then*

$$\bar{x}_{l_i} = v_i. \tag{4$'$}$$

It is possible to give an extension of this corollary that will be useful for us later, namely:

COROLLARY 2. *Under the assumptions of Corollary* 1 *if at least one of the following two conditions hold*

$$\sum_{i=0}^{r_1+1} m_i = m + \sum_{i=1}^{s_1 - 1} (n_i + \sigma_i) + 1, \qquad m_{r_1+1} + n_{s_1} \geqslant m + 1, \quad \sigma_{s_1} = 1 \tag{5}$$

$$\sum_{i=0}^{r_1} m_i = n_{s_1} + \sum_{i=1}^{s_1 - 1} (n_i + \sigma_i), \qquad m_{r_1+1} + n_{s_1} \geqslant m + 1, \quad \sigma_{s_1} = 1 \tag{6}$$

*then*

$$\bar{x}_{l_{s_1}} = v_{s_1}. \tag{4)''$$

*Proof.* If in the sequence $\bar{x}_1 \leqslant \bar{x}_2 \leqslant \cdots \leqslant \bar{x}_{N_1}$, $x_{r_1+1}$ appears in the places $j+1,\ j+2, ..., j+m_{r_1+1}$:

$$(5) \Rightarrow j+m_{r_1+1} = m+l_{s_1-1}+1,$$
$$(6) \Rightarrow j+1 = l_{s_1}.$$

With $\sigma_{s_1} = 1$, $(m+l_{s_1-1}+1)-l_{s_1}+1 = m+1-n_{s_1}$, but since $m_{r_1+1} \geqslant m+1-n_{s_1}$ this implies $\bar{x}_{l_{s_1}} = \bar{x}_{m+l_{s_1-1}+1}$ in both cases. Hence by (4), (4)″ follows. ∎

LEMMA 2. *For a given $a \in (0,1)$, and a given pair of multiplicities $m_{r_1+1}+n_{s_1} \geqslant m+1$, consider the class of all generalized monosplines with a zero of multiplicity $m_{r_1+1}$ at a and a knot of multiplicity $n_{s_1}$ at a, and with $\sigma_{s_1}=1$. (We call such a point a, a break point).*

*For any F in this class with a full set of zeros, let $F_1 := F$ on $[0, a]$, $F_2 := F$ on $[a, 1]$. Then if q is the multiplicity of the zero of $F_1$ at a, and p is the multiplicity of the zero of $F_2$ at a, either* I *or* II *holds.*

I. (a) $q = m_{r_1+1}-1,\ p = m-n_{s_1}$.

(b) *$F_1$ has a full set of zeros on $[0, a]$ and its parameters satisfy*

$$\sum_{i=0}^{r_1+1} m_i - 1 = m + \sum_{i=1}^{s_1-1} (n_i+\sigma_i). \tag{7}$$

(c) *$F_2$ has a full set of zeros on $[a, 1]$ and its parameters satisfy*

$$(m-n_{s_1}) + \sum_{i=r_1+2}^{r+1} m_i = m + \sum_{i=s_1+1}^{s} (n_i+\sigma_i). \tag{8)'$$

II. (a) $q = m-n_{s_1},\ p = m_{r_1+1}-1$.

(b) *$F_1$ has a full set of zeros on $[0, a]$ and its parameters satisfy*

$$\sum_{i=0}^{r_1} m_i + (m-n_{s_1}) = m + \sum_{i=1}^{s_1-1} (n_i+\sigma_i). \tag{8}$$

(c) *$F_2$ has a full set of zeros in $[a, 1]$ and its parameters satisfy*

$$(m_{r_1+1}-1) + \sum_{i=r_1+2}^{r+1} m_i = m + \sum_{i=s_1+1}^{s} (n_i+\sigma_i). \tag{7)'$$

*Conversely consider any pair $F_1$ defined on $[0, a]$, $F_2$ defined on $[a, 1]$, which satisfy either* I *or* II, *and let*

$$F = \begin{cases} F_1 & \text{on } [0, a] \\ \sum_{j=1}^{n_{s_1}} b_j (x-a)_+^{m-j} + F_2 & \text{on } [a, 1], \end{cases} \tag{9}$$

*where $F_1 + \sum_{j=1}^{n_{s_1}} b_j (x-a)_+^{m-j} \equiv 0$, $x \geqslant a$. Then $F$ is a member of the class and has a full set of zeros over $[0, 1]$.*

*Proof.* The Budan–Fourier theorem implies:

For $[0, a]$

$$\sum_{i=0}^{r_1} m_i \leqslant \sum_{i=1}^{s_1-1} (n_i + \sigma_i) + (m - S^+ LF(a)) \tag{10}$$

and for $[a, 1]$

$$\sum_{i=r_1+2}^{r+1} m_i \leqslant \sum_{i=s_1+1}^{s} (n_i + \sigma_i) + (m - S^+ RF(a)). \tag{11}$$

Combining we get

$$\sum_{\substack{i \neq r_1+1 \\ i=0}}^{r+1} m_i \leqslant \sum_{\substack{i \neq s_1 \\ i=1}}^{s} (n_i + \sigma_i) + (m - S^+ LF(a) - S^+ RF(a)) + m. \tag{12}$$

On the other hand, since $F$ has a full set of zeros and $\sigma_{s_1} = 1$, we have (see Schumaker [11, Formula 8.62])

$$\gamma_a = S^+ LF(a) + S^+ RF(a) + n_{s_1} + 1 - m_{r_1+1} = 0$$

or

$$m - S^+ LF(a) - S^+ RF(a) = (n_{s_1} + 1) - m_{r_1+1}. \tag{13}$$

By (3) and (13), (12) is an equality and so are (10) and (11).

We can also write (13) as

$$S^+ LF(a) + S^+ FR(a) = (m - n_{s_1}) + (m_{r_1+1} - 1). \tag{13$'$}$$

By assumption $m_{r_1+1} - 1 \geqslant m - n_{s_1}$. Since the zero in $[0, 1]$ is of order $m_{r_1+1}$, we need either $q$ or $p$ to be at least of order $m_{r_1+1} - 1$. We consider three cases.

*Case* 1. $q = m - n_{s_1}$. Then $m_{r_1+1} - 1 \geqslant m - n_{s_1}$ implies (by considering $>$ and $=$ separately)

$$\begin{aligned} S^+LF(a) &\geqslant m - n_{s_1} \\ p &\geqslant m_{r_1+1} - 1 \\ S^+RF(a) &\geqslant (m_{r_1+1} - 1). \end{aligned} \tag{14}$$

Combined with (13)′ this implies all the equations in (14) are equalities. Since (10) and (11) are equalities, this proves II, including Eqs. (7)′ and (8).

*Case* 2. $q < m - n_{s_1}$. This is impossible for then the zero in [0, 1] is of order $q < m_{r_1+1}$.

*Case* 3. $q > m - n_{s_1}$. Since the knot is of order $n_{s_1}, p = m - n_{s_1}$, and hence

$$\begin{aligned} S^+RF(a) &\geqslant m - n_{s_1} \\ q &\geqslant m_{r_1+1} - 1 \\ S^+LF(a) &\geqslant m_{r_1+1} - 1. \end{aligned} \tag{15}$$

Combined with (13)′ this implies all the equations in (15) are equalities. Since (10) and (11) are equalities this proves I, including Eqs. (7) and (8)′.

This completes the proof of the first part of the lemma. For the second part:

*Case* 1. Conversely if $F_1$ and $F_2$ satisfy the stated conditions II, then $F$ defined by (9) satisfies (3) because if $\sigma_{s_1} = 1$, then it follows from (14) (with all equalities) that $(-1)^q \operatorname{sgn} D_-^q F_1(a) = (-1)^{p+1} \operatorname{sgn} D_+^p F_2(a)$, which implies the zero at $a$ is of order $p + 1$.

*Case* 2. Conversely, if $F_1$ and $F_2$ satisfy the stated conditions I, then $F$ defined by (9) satisfies (3) because if $\sigma_{s_1} = 1$, then it follows from (15) (with all equalities) that $-\operatorname{sgn} D_-^q F_1(a) = \operatorname{sgn} D_+^p F_2(a)$, which implies the zero is of order $q + 1$.

We define an indecomposable problem of Type II as a problem with a full set of zeros, i.e., it satisfies (16), for some $\bar{r}$ and $\bar{s}$,

$$\sum_{i=0}^{\bar{s}+1} \bar{m}_i = m + \sum_{j=1}^{\bar{r}} (\bar{n}_j + \sigma_j) \tag{16}$$

with all $\sigma_j = 1$, and with all $\bar{n}_j < m$, and if a knot with multiplicity $\bar{n}_k$ and a zero with multiplicity $\bar{m}_g$ coalesce then $\bar{m}_g + \bar{n}_k \leqslant m$.

We define an indecomposable problem of Type I as a problem with a full set of zeros, i.e., it satisfies (17), for some $\bar{r}$ and $\bar{s}$,

$$\sum_{j=0}^{\bar{r}+1} m_j = m + \sum_{i=1}^{\bar{s}} (n_i + \sigma_i) \tag{17}$$

with all $\sigma_i = 0$. ▮

LEMMA 3. *Any problem of Type* II (*i.e., all* $\sigma_i = 1$) *can be decomposed into* $h+1$ *indecomposable problems with break points at* $h$ *zeros* $\tilde{y}_k$, $k = 1, ..., h$; (*$h$ and the particular zeros, depending on the problem*), *such that any solution of the original problem is obtained by piecing together solutions of the indecomposable problems. Thus the original problem has a solution if and only if the individual indecomposable problems have a solution.*

*Proof.* It follows from Lemma 2 and Corollary 2 that a necessary and sufficient condition for a knot or zero to be a break point is that Eqs. (5) or (6) applied to problems of Type II hold. It follows from Lemma 2 that if a knot and zero coalesce at a point $a$ with their sum $\geqslant m+1$, that point is a break point. A special case is when $\bar{n}_k = m$, for then (4)′ implies that $t_k$ is also a zero and hence a break point.

Thus if for every set of parameters such that

$$\sum_{i=0}^{s_1+1} \bar{m}_i = m + \sum_{i=1}^{r_1-1} (\bar{n}_i + 1) + 1, \qquad \bar{m}_{s_1+1} + \bar{n}_{r_1} \geqslant m+1 \tag{5$''$}$$

or

$$\sum_{i=0}^{s_1} \bar{m}_i = \bar{n}_{r_1} + \sum_{i=1}^{r_1-1} (\bar{n}_i + 1), \qquad \bar{m}_{s_1+1} + \bar{n}_{r_1} \geqslant m+1 \tag{6$''$}$$

if we choose $y_{s_1+1}$ as a break point, Lemma 3 follows. ▮

LEMMA 4. *Consider the following* 1–1 *transformation of problems of Type* II *into a subset of problems of Type* I:

*For* $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$:

$$\begin{aligned}
&v = 1 - y, \qquad x = 1 - t, \qquad i' = s + 1 - i, \qquad j' = r + 1 - j,\\
&n_i + \sigma_i = \bar{m}_{i'} \qquad \textit{with } \sigma_i = 1 \textit{ at break points } y_i, \sigma_i = 0 \textit{ otherwise.}\\
&m_j = \bar{n}_{j'} + 1,\\
&v_i = 1 - y_{i'}, \qquad x_j = 1 - t_{j'} \qquad (\textit{also for } i = 0, s+1; j = 0, r+1)\\
&m_{r+1} = \bar{n}_0; \qquad \bar{n}_{r+1} = m_0; \qquad n_0 = \bar{m}_{s+1}; \qquad \bar{m}_0 = n_{s+1};\\
&\bar{m}_0 + \bar{n}_0 = m_0 + n_0 = m = n_{s+1} + m_{r+1} = \bar{n}_{r+1} + \bar{m}_{s+1}.
\end{aligned} \tag{18}$$

A. *If the problem of Type* II *satisfies* (16), *the problem of Type* I *induced by* (18) *satisfies* (17).

B. *If a zero* $y_{s_1+1}$ *in a problem of Type* II *is a break point, the knot* $v_{s-s_1}$ *is a break point in the corresponding problem of Type* I.

C. *Indecomposable problems go into indecomposable problems in the sense that if the solution of the problem of Type* II *is obtained by piecing together* $h+1$ *solutions of indecomposable problems, the solution of the corresponding problem of Type* I *is obtained by piecing together* $h+1$ *solutions of indecomposable problems. In particular, the solution to the problem of Type* I *exists if and only if the* $h+1$ *individual indecomposable problems have solutions.*

*Proof of* A.

$$\sum_{i=0}^{s+1} \bar{m}_i = \sum_{i=1}^{s} (n_i + \sigma_i) + (m - m_{r+1}) + m - m_0;$$

$$m + \sum_{j=1}^{r} (\bar{n}_j + 1) = m + \sum_{j=1}^{r} m_j$$

so A follows. ▮

*Proof of* B. Say, for example, Eq. (5)″ is satisfied, i.e., $y_{s_1+1}$ is a break point, satisfying (5)″. Then $\bar{m}_{s_1+1} + \bar{n}_{r_1} = n_{s-s_1} + m_{r+1-r_1} \geqslant m+1$ and $\sigma_{s-s_1} = 1$, thus

$$\sum_{i=0}^{s_1+1} \bar{m}_i = \sum_{i=s-s_1}^{s} (n_i + \sigma_i) + (m - m_{r+1})$$

$$m + \sum_{j=1}^{r_1-1} (\bar{n}_i + 1) + 1 = m + \sum_{j=(r-r_1)+2}^{r} m_j + 1.$$

Hence

$$(m - n_{s-s_1}) + \sum_{j=(r-r_1)+2}^{r+1} m_j = m + \sum_{i=(s-s_1)+1}^{s} (n_i + \sigma_i).$$

Thus (8)′ holds and since A holds also (7) of Lemma 2 holds. Hence by Lemma 2, $v_{s-s_1}$ is a break point. A similar argument applies if Eq. (6)″ is satisfied. ▮

*Proof of* C. Since no other knots $v_i$ of the problem of Type I except those described in B have been assigned $\sigma_i = 1$, and since by Lemma 2 at each break point $v_{s-s_1}$ we can break the original problem of Type I into two problems both with a full set of zeros, it follows that indecomposable problems go into indecomposable problems. ▮

It follows from Lemmas 3 and 4 that we may restrict our discussion to solutions of indecomposable problems. To simplify the discussion and without loss of generality let us assume that the indecomposable problem involves the interval [0, 1]; i.e., let us imagine that the original problems (1), (17) (with all $\sigma_i = 0$) and (2), (16) are indecomposable problems. We introduce the following notation: we let $\bar{y}_1 \leqslant \bar{y}_2 \leqslant \cdots \leqslant \bar{y}_N$ be the sequence obtained from $y_1, ..., y_{s+1}$ by repeating $y_i$, $\bar{m}_i$ times, $i = 1, ..., s+1$, and let $\bar{t}_1 \leqslant \bar{t}_2 \leqslant \cdots \leqslant \bar{t}_N$ be the sequence obtained from $t_0, t_1, ..., t_r$ by repeating $t_0$, $\bar{n}_0$ times and $t_i$, $\bar{n}_i + 1$ times, $i = 1, ..., r$. We let $\bar{x}_1 \leqslant \bar{x}_2 \leqslant \cdots \leqslant \bar{x}_N$ be the sequence obtained from $x_1, ..., x_{r+1}$ by repeating $x_i$, $m_i$ times ($i = 1, ..., r+1$), and let $\bar{v}_1 \leqslant \bar{v}_2 \leqslant \cdots \leqslant \bar{v}_N$ be the sequence obtained from $v_0, v_1, ..., v_s$ by repeating $v_i$, $n_i$ times, $i = 0, ..., s$. We consider furthermore two sets of points. For given $\bar{y}_i$, 1, ..., $N$, let $\tilde{D}$ be the set of all $\bar{t}_i$ defined as above such that

$$\bar{y}_{i-m} < \bar{t}_i < \bar{y}_i \tag{19}$$

when the indices are meaningful.

For given $v_i$, $i = 1, ..., N$, let $D$ be the set of all $\bar{x}_i$ defined as above such that

$$\bar{v}_i < \bar{x}_i < \bar{v}_{i+m} \tag{20}$$

when the indices are meaningful.

Since we are dealing with indecomposable problems, it follows from Corollary 1 (applied to $M(y)$) that if a problem of Type II is indecomposable and has a solution, the corresponding $\bar{t}_i$ belong to $\tilde{D}$; i.e., $\tilde{D}$ is a non-empty open set. Moreover it follows from Corollary 1 that if $F(x^*, x)$ has a full set of zeros then the corresponding $\bar{x}_i$ must belong to the closure of $D$. Thus if a problem of Type I has a solution, the corresponding $\bar{x}_i$ will belong to the closure of $D$.

The map (18) takes the set $\tilde{D}$ associated with an indecomposable problem of Type II, onto the corresponding set $D$ associated with the corresponding indecomposable problem of Type I. A proper problem of Type II is one such that the sets $\tilde{D}$, associated with the indecomposable problems that the original problem can be decomposed into, are all non-empty. As noted above for a problem of Type II to have a solution it is necessary that it be a proper problem. Later we show it is sufficient.

We will restrict our investigation from now on to what we define as well-posed $L_1$ problems. They are defined as problems of Type I, obtained from proper problems of Type II by the transformation (18).

THEOREM 1. *Every well-posed $L_1$ problem has at least one solution.*

*Proof.* By our previous remarks it is sufficient to prove this result for indecomposable problems. Thus we will prove the result for Eq. (1), with all $\sigma_i = 0$, under the proviso that $D$ is non-empty.

Let

$$g_0 = 0, \qquad g_i = g_{i-1} + n_i, \qquad i = 1, ..., s.$$

Then

$$u_{g_i+j}(x) = \Phi_m^{j-1}(x, v_i) \begin{cases} i = 0 \cdots s, \\ j = 1 ..., n_i, \end{cases} \qquad \text{defines } u_1, ..., u_N. \tag{21}$$

We define $\Phi_m\binom{\bar{x}_1, ..., \bar{x}_N}{\bar{v}_1, ..., \bar{v}_N}$ as does Micchelli [10, Eq. (16)], and note that for points in $D$, $\Phi_m\binom{\bar{x}_1, ..., \bar{x}_N}{\bar{v}_1, ..., \bar{v}_N} > 0$ (see Schumaker [11, Theorem 4.78]).

Set

$$u_{N+1}(x) = \sum_{i=0}^{s} (-1)^{g_i} \int_{v_i}^{v_{i+1}} \Phi_m(x, v)\, dv = G(x). \tag{22}$$

In $D$ let $x^* = (x_1 \cdots x_r)$

$$\begin{aligned} F(x^*, x) &= \frac{\bigcup \begin{pmatrix} u_1 \cdots u_N u_{N+1} \\ \bar{x}_1 \cdots \bar{x}_N x \end{pmatrix}}{\bigcup \begin{pmatrix} u_1 \cdots u_N \\ \bar{x}_1 \cdots \bar{x}_N \end{pmatrix}}; \qquad (23) \\ &= G(x) + \sum_{i=1}^{N} b_i(x^*)\, u_i(x) \\ &= G(x) + \sum_{i=0}^{s} \sum_{j=0}^{n_i - 1} b_{ij}\, \Phi_m^j(x, v_i), \end{aligned}$$

where

$$\bigcup \begin{pmatrix} u_1 \cdots u_N \\ \bar{x}_1 \cdots \bar{x}_N \end{pmatrix} = \Phi_m \begin{pmatrix} \bar{x}_1, ..., \bar{x}_N \\ \bar{v}_1, ..., \bar{v}_N \end{pmatrix} = \det\{u_i(\bar{x}_j);\, i, j = 1, ..., N\}$$

with the usual convention in case of coincidence among the $\bar{x}_i$'s (see Schumaker [11, Sect. 4.10]). Let

$$Q(x^*) = \int_0^1 |F(x^*, x)|\, dx.$$

Then we prove the following:

THEOREM 1'. $\min_{x^* \in D} Q(x^*)$ *is attained for some* $x^* \in D$.

THEOREM 2′. *At the minimum of $Q(x^*)$, $x^* = (x_1, ..., x_r)$, we have*

$$0 = x_0 < x_1 < x_2 < \cdots < x_r < x_{r+1} = 1.$$

Theorem 1 will then follow. ▮

We prove Theorem 1′ and Theorem 2′ below.

THEOREM 2. *If a well-posed $L_1$ problem has a solution with zeros $0 < x_1 < \cdots < x_r < x_{r+1} = 1$, the proper problem of Type* II *that it comes from (by the transformation* (18)), *has a solution with knots $t_j = 1 - x_{r+1-j}, j = 1 \cdots r$.*

*Proof.* Once again it is sufficient to prove this result for indecomposable problems. Thus we assume that $F(x^*, x)$ of Eq. (1), with all $\sigma_i = 0$, is a minimum for a Type I problem, and that $D$ is non-empty. Assuming Theorem 1′ and Theorem 2′ proved, and that $Q(x^*)$ attains its minimum at $x^* = (x_1, ..., x_r), 0 < x_1 < \cdots < x_r < 1$ in $D$, then we assert that there are constants $\bar{a}_j$ and $\bar{a}_{ij}$ such that

$$\int_0^1 (\operatorname{sgn} F)\, u_v(x)\, dx = \sum_{j=0}^{m_{r+1}-1} \bar{a}_j u_v^j(1) + \sum_{i=1}^{r} \sum_{j=0}^{m_i-2} \bar{a}_{ij}\, u_v^j(x_i), \qquad v = 1, ..., N \tag{24}$$

[2]

To establish this, first assume $x_k$ is a zero of order $q$, and a knot of order $w$ with $q + w < m - 2$. Then clearly

$$\frac{\partial Q(x^*)}{\partial x_k} = \int_0^1 (\operatorname{sgn} F) \frac{\partial F}{\partial x_k}\, dx = 0 \tag{25$'$}$$

$$\frac{\partial F}{\partial x_k} \text{ is a linear combination of } u_i(x), \qquad i = 1 \cdots N \tag{25$''$}$$

and

$$\frac{d^j}{dx^j} \frac{\partial F}{\partial x_k}(x^*, x)\bigg|_{x = x_l} = \begin{cases} 0, & j = 0 \cdots m_l - 1, \quad l \neq k \\ 0, & j = 0 \cdots m_l - 2, \quad l = k \\ \neq 0, & j = m_k - 1, \quad l = k \end{cases} \tag{25$'''$}$$

(see Remark 1 of [3]).

For the general case where we can only assert $w + q \leqslant m$, we will demonstrate that there still exists a function, call it $F_k$, that satisfies (25)′. (25)″ and (25)‴. (See Lemmas 12 and 13.)

[2] If $m_i = 1$, then $\bar{a}_{ij} = 0$ for all $j$.

Since we are in $D$, we are assured of the existence of $\bar{a}_j$, $\bar{a}_{ij}$ such that (24) holds, but with the second sum being $\sum_{i=1}^{r} \sum_{j=0}^{m_i-1} \bar{a}_{ij} u_v^j(x_i)$. However, it follows if (25)', (25)'', and (25)''' hold for some $k$ that $\bar{a}_{k,m_k-1} = 0$. Hence (24) follows.

Finally, performing the transformation (18) on the $u_v(x)$, $v = 1 \cdots N$, Eq. (24) is equivalent to the statement that the $M(y)$ of Eq. (2) has a zero of order $\bar{m}_i$, at $y_i$, $i = 1, ..., s+1$. ∎

THEOREM 3. *Every well-posed $L_1$ decomposable problem and every proper problem of Type* II *have one and only one solution.*

*Proof.* Once again it is sufficient to prove the result for indecomposable problems. We have already proved the existence of solutions. We now establish uniqueness. In the proof of Theorem 2, we showed that if the indecomposable $L_1$ problem has a solution, then the quadrature formula (24) with points $x_1 < x_2 \cdots < x_r$ follows, and that (24) is equivalent to the solution of the corresponding problem of Type II. Hence, if either the $L_1$ indecomposable problem or the indecomposable problem of Type II had two solutions this would imply that the quadrature formula (24) had a second solution with points $x_1' < x_2' < \cdots < x_r'$.

If we apply the Gaussian transform to the functions $u_i(x)$, $i = 1, ..., N$, we obtain $u_i(\varepsilon, x)$, $i = 1, ..., N$, which is an Extended Chebyshev System. If we also use the implicit function theorem on (24) (with parameters $\bar{a}_j$, $\bar{a}_{ij}$, and $x_1, ..., x_r$) the Jacobian is $\prod_{i=1}^{r} \bar{a}_{i,m_i-2} \Phi_m \binom{\bar{x}_1, ..., \bar{x}_N}{\bar{v}_1, ..., \bar{v}_N}$ which is non-zero, since first we are in $D$ and second if any of the $\bar{a}_{i,m_i-2} = 0$,[3] it would follow from the Budan–Fourier Theorem that the corresponding $M(y)$ would not have enough zeros. Thus if (24) had two different solutions $x_1 < \cdots < x_r$ and $x_1' < x_2' < \cdots < x_r'$, it would follow that the resulting Generalized Gaussian Quadrature Formula for the $u_i(\varepsilon, x)$, $\varepsilon > 0$, had two sets of canonical points, which is a contradiction (see Braess [5, Chap. I, 4.2] and Bojanov *et al.* [4]). This establishes uniqueness. ∎

*Proof of Theorem* 1'. The proof of Theorem 1' is long and technical. The position of the knots $v_i$, $i = 1, ..., s$, are fixed. We consider $Q(x^*)$ for $x^* = (x_1, ..., x_r) \in D$. We show that if $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_r)$ belongs to the boundary of $D$, then $Q(\tilde{x})$ is not the minimum of $Q(x^*)$. To avoid some cumbersome notation we will only consider two cases, and we will restrict ourselves to the situation where all $m_i$ are even, thus $F \geqslant 0$. It will be clear that the general situation can be treated by our method. We consider $x_k$ to be a zero of multiplicity $m_k$, $v_f$ to be a knot of multiplicity $n_f$, with $m_k + n_f \geqslant m + 1$, and $x_k \to v_f$. In Lemmas 5, 6 and 7 we treat the case where $x_k > v_f$, and in Lemmas 8, 9, 10, and 11 we treat the case $x_k < v_f$.

[3] If $m_i = 1$, replace $\bar{a}_{i,m_i-2}$ by $\pm 2$.

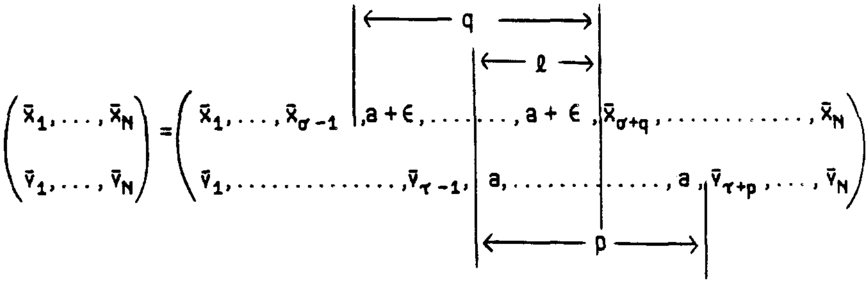


FIG. 1. Overlap $l$ at $\varepsilon=0$.

We begin with

DEFINITION 1. (See Fig. 1). Assume $F(x^*, x)$ has a zero of multiplicity $q$ at $x_k$, with $\bar{x}_{\sigma+i}=x_k$, $i=0, 1, ..., q-1$. Further assume $v_f=a$ is a knot of multiplicity $p$ with $\bar{v}_{\tau+j}=v_f=a$, $j=0, ..., p-1$. We say this is an overlap $l$ at $\varepsilon=0$ if $\sigma+q=\tau+l$; i.e., if $x_k=a$, then $\bar{x}_{\tau+j}=\bar{v}_{\tau+j}=a, j=0, ..., l-1$. Further $\bar{v}_{\tau-1}<\bar{v}_\tau<\bar{v}_{\tau+p}$; $\bar{x}_{\sigma-1}<\bar{x}_\sigma<\bar{x}_{\sigma+q}$; $0\leqslant l\leqslant \min(p, q)$. Note that if $l=0$, then even for $\varepsilon=0$, $x^*\in D$.

LEMMA 5. *Assume, in D, that $F(x^*, x)$ has a zero of multiplicity $q$ at $x_k=a+\varepsilon$, and a knot of multiplicity $p$ at $v_f=a$, with overlap $l$ at $\varepsilon=0$. Keeping all other zeros (and, of course knots) fixed, then as a function of $\varepsilon$*

$$\begin{aligned}\Phi_N=\Phi_m&\begin{pmatrix}\bar{x}_1, ..., \bar{x}_N\\ \bar{v}_1, ..., \bar{v}_N\end{pmatrix}\\ &=a_z\varepsilon^z+\text{higher order terms}\\ &\quad a_z\neq 0, \qquad z=l[(m+l)-(p+q)]\geqslant 0.\end{aligned}$$

*Proof.* Since $\bar{x}_\sigma>\bar{v}_{\tau+p-1}$, $\bar{x}_\sigma<\bar{v}_{\sigma+m}$ we conclude $\tau+p-1<\sigma+m$ and hence $q+p-1<m+l$. Thus $z\geqslant 0$ as stated.

If Mat $\Phi_N$ is the matrix of which $\Phi_N$ is the determinant then

$$\text{Mat}\,\Phi_N=\begin{array}{c} \\ \sigma-1\\ q\\ v\end{array}\begin{array}{c}\begin{array}{ccc}\tau-1 & p & u\end{array}\\ \begin{pmatrix}A & 0 & 0\\ B & L & 0\\ C & E & K\end{pmatrix}\end{array}$$

with $\tau-1+p+u=\sigma-1+q+v=N$.

The elements of $A$, $C$, $E$, $K$ are constants. The elements of $B$ are polynomials in $\varepsilon$; and the elements of $L$ are

$$L_{\alpha\beta} = ((-1)^{\beta-1} \varepsilon_*^{m-(\alpha+\beta)+1}), \qquad \alpha = 1, ..., q \qquad \beta = 1, ..., p.$$

$$\varepsilon_*^j = \begin{cases} \varepsilon^j/j!, & j \geqslant 1 \\ 1, & j = 0 \\ 0, & j < 0. \end{cases}$$

Let the columns of $A$ be $a_1, ..., a_{\tau-1}$, those of $L$ $f_1, ..., f_p$, etc. Using Laplace's expansion on the first $(\sigma-1)+q$ rows we find

$$\Phi_N = \sum_{\substack{\theta+w=(\sigma-1)+q \\ w \geqslant l}} \pm \det \begin{pmatrix} a_{i_1}, ..., a_{i_\theta}, 0, ..., 0 \\ b_{i_1}, ..., b_{i_\theta}, f_{j_1}, ..., f_{j_w} \end{pmatrix}$$
$$\times \text{det of complement}$$

(since $\sigma - 1 + q = \tau - 1 + l$, if $w < l$, then $\theta > \tau$, which is impossible). Using Laplace's expansion again, this time on the last $w$ columns, with $L\binom{h_1, ..., h_w}{j_1, ..., j_w}$ a $w \times w$ subdeterminant of $L$ $(w \geqslant l)$

$$\det \begin{pmatrix} a_{i_1}, ..., a_{i_\theta}, 0, ..., 0 \\ b_{i_1}, ..., b_{i_\theta}, f_{j_1}, ..., f_{j_w} \end{pmatrix} = \sum_{h_1 < h_2 < \cdots < h_w \leqslant q} L \begin{pmatrix} h_1, ..., h_w \\ j_1, ..., j_w \end{pmatrix}$$
$$\times \text{det of complement.}$$

It readily follows using the usual definition of determinants that

$$L \begin{pmatrix} h_1, ..., h_w \\ j_1, ..., j_w \end{pmatrix} = C\varepsilon^{w(m+1)-(h_1+h_2+\cdots+h_w)-(j_1+\cdots+j_w)},$$

where $C$ is the sum of the coefficients over all paths where $h_a + j_b \leqslant m+1$ for all $L_{h_a, j_b}$ in the path. Since

$$L \begin{pmatrix} p-l+1, ..., p \\ q-l+1, ..., q \end{pmatrix} = C_1 \varepsilon^{l[(m+l)-(p+q)]}, \qquad C_1 \neq 0,$$

as is easily calculated [see Eq. (29)] we see it contains the lowest power of $\varepsilon$ of any $L\binom{h_1, ..., h_w}{j_1, ..., j_w}$, $w \geqslant l$.

The terms multiplying $L\binom{p-l+1, ..., p}{q-l+1, ..., q}$ in the expansion of $\Phi_N$ consist of two determinants. The first is

$A_1^{\tau-1,\tau-1}$ whose first $\sigma-1$ rows are those of $A$, and whose last $q-l$ rows are the first $q-l$ rows of $B$,

and

$A_2^{v,v}$ whose first $(p-l)$ columns are the first $(p-l)$ columns of $E$, and whose last $u$ columns are those of $K$.

If $\Phi_m\binom{\bar{x}_1,\ldots,\bar{x}_N}{\bar{v}_1,\ldots,\bar{v}_N}>0$ and you remove $\bar{x}_j$ and $\bar{v}_j$ for some $j$, it follows from $\bar{v}_i<\bar{x}_i<\bar{v}_{i+m}$ all $i$, that $\Phi_m\binom{\bar{x}_1,\ldots,\bar{x}_{j-1},\bar{x}_{j+1},\ldots,\bar{x}_N}{\bar{v}_1,\ldots,\bar{v}_{j-1},\bar{v}_{j+1},\ldots,\bar{v}_N}>0$. Hence since both $A_1^{\tau-1,\tau-1}$ and $A_2^{v,v}$ are obtained from $\Phi_m\binom{\bar{x}_1,\ldots,\bar{x}_N}{\bar{v}_1,\ldots,\bar{v}_N}$ by deleting corresponding rows and columns, neither is zero. Further since $\bar{v}_u\leqslant\bar{v}_{\tau-1}<a\leqslant a+\varepsilon=\bar{x}_u$, $u=\sigma,\sigma+1,\ldots,\tau-1$, it follows that $A_1^{\tau-1,\tau-1}$ is not zero even for $\varepsilon=0$. Thus the lemma is extablished. ▮

LEMMA 6. *Assume in D, with all $m_i$ even, that $F(x^*,x)$ has a zero of order $q$ at $x_k=a+\varepsilon$, and that it has a knot at $v_f=a$ of multiplicity $p$, with overlap $l$ where $m+l>p+q$ $l\geqslant 1$. Then* $\lim_{\varepsilon\downarrow 0} b_{f,n_f-1}=\infty$ [*see* (23)].

*Proof.* Using the notation of Definition 1 and Lemma 5

$$b_{f,n_f-1}=b_{\tau+p-1}(x^*)=\frac{U\begin{pmatrix}u_1,\ldots,\hat{u}_{\tau+p-1},\ldots,u_{N+1}\\ \bar{x}_1,\ldots\ldots\ldots\ldots,\bar{x}_N\end{pmatrix}}{U\begin{pmatrix}u_1,\ldots,u_N\\ \bar{x}_1,\ldots,\bar{x}_N\end{pmatrix}}$$

(where $\hat{u}_i$ signifies the term $u_i$ does not appear)

$$=\frac{\sum_{i=0}^{s}(-1)^{g_i}\int_{v_i}^{v_{i+1}}\Phi_m\begin{pmatrix}\bar{x}_1,\ldots\ldots\ldots\ldots\ldots\ldots,\bar{x}_N\\ \bar{v}_1,\ldots,\bar{v}_{g_f-1},v,\bar{v}_{g_f+1},\ldots,v_N\end{pmatrix}dv}{\Phi_m\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_N\\ \bar{v}_1,\ldots,\bar{v}_N\end{pmatrix}}$$

Let $U_i$ be the $N$ tuple

$$(\overbrace{v_1,\ldots,v_1}^{n_1},\ldots,\overbrace{v_f,\ldots,v_f}^{n_f-1},\ldots,\overbrace{v_s,\ldots,v_s}^{n_s},v)$$

rearranged in increasing order with $v$ between the last $v_i$ and the first $v_{i+1}$. Then

$$b_{f,n_f-1}=\pm\frac{\sum_{i=0}^{f-1}\int_{v_i}^{v_{i+1}}\Phi_m\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_N\\ U_i\end{pmatrix}dv-\sum_{i=f}^{s}\int_{v_i}^{v_{i+1}}\Phi_m\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_N\\ U_i\end{pmatrix}dv}{\Phi_m\begin{pmatrix}\bar{x}_1,\ldots,\bar{x}_N\\ \bar{v}_1,\ldots,\bar{v}_N\end{pmatrix}}$$

We note that $p$ decreases by 1 in all terms in the numerator, while $l$ decreases by 1 in the first sum, but not in the second. (If $p=l$, the term $\int_{v_f}^{v_{f+1}}\Phi_m\binom{\bar{x}_1,\ldots,\bar{x}_N}{U_f}dv$ has $p$ decreasing by 1 and $l$ decreasing by 1; but $\Phi_m\binom{\bar{x}_i,\ldots,\bar{x}_N}{U_f}=0$ for $v>a+\varepsilon$, and hence this term is negligible compared to the term $f-1$ in the first series.) Thus we can assert that

$$b_{f,n_f-1}=\frac{\sum_{i=0}^{f-1} a_i(\varepsilon)\,\varepsilon^{(l-1)[(m+l)-(p+q)]}-\sum_{i=f}^{s} b_i(\varepsilon)\,\varepsilon^{l[(m+l+1)-(p+q)]}}{c(\varepsilon)\,\varepsilon^{l[(m+l)-(p+q)]}}$$

$$=\sum_{i=0}^{f-1}\frac{a_i(\varepsilon)}{c(\varepsilon)}\varepsilon^{-[(m+l)-(p+q)]}-\sum_{i=f}^{s}\frac{b_i(\varepsilon)}{c(\varepsilon)}\varepsilon^{l}$$

with

$$\begin{aligned} a_i(\varepsilon)\,a_{i+1}(\varepsilon)&\geqslant 0, \qquad i=1\cdots f-2\\ b_i(\varepsilon)\,b_{i+1}(\varepsilon)&\geqslant 0, \qquad i=f\cdots s-1\\ a_{f-1}(0)&\neq 0\\ c(0)&\neq 0. \end{aligned}$$

Thus the conclusion follows. ∎

LEMMA 7. *Assume in D, with all $m_i$ even, that $F(x^*, x)$ has a zero of order $q=m_k$ at $x_k=a+\varepsilon$, and that it has a knot at $v_f=a$ of multiplicity $p=n_f$, with overlap $l\geqslant 1$ where $m+l=p+q$. Keeping all other zeros fixed, and setting $F(\varepsilon, x):=F(x^*, x)$ and*

$$Q(\varepsilon)=\int_0^1 F(\varepsilon, x)\,dx$$

*we have for sufficiently small positive $\varepsilon$:*

$$\frac{\partial Q(\varepsilon)}{\partial\varepsilon}<0.$$

*Hence $F(\varepsilon, x)$ is not a minimizing sequence.*

*Proof.* We will show the following, for $\varepsilon>0$

(a) $$\frac{d^j}{dx^j}\frac{\partial F(x^*, x)}{\partial x_k}\bigg|_{x=x_k}=0, \qquad j\leqslant m_k-2;$$

$$\frac{d^{m_k-1}}{dx^{m_k-1}}\frac{\partial F(x^*, x)}{\partial x_k}\bigg|_{x=x_k}=(-1)\frac{d^{m_k}}{dx^{m_k}}F(x^*, x)\bigg|_{x=x_k}=-F^{m_k}\neq 0;$$

(b) $$\frac{\partial F(x^*, x)}{\partial x_k}\begin{cases}\geqslant 0, & x<x_k\\ \leqslant 0, & x>x_k;\end{cases}$$

(c) If $$\frac{\partial F(x^*, x)}{\partial x_k}=\sum_{i=0}^{s}\sum_{j=0}^{n_i-1} c_{ij}(x-v_i)_+^{m-1-j} \quad c_{ij}=O(\varepsilon)\,F^{m_k} \quad \text{if } i<f.$$

The conclusion follows from (a), (b), and (c). To establish this we set $\partial F(x^*, x)/\partial x_k=F^{m_k}H(\varepsilon, x)$. Then $H(\varepsilon, x)$ is continuous in $\varepsilon$, and $H(0, x)$ is well defined. Further by (c) $\int_0^{x_k} H(0, x)\,dx=0$, and by (a) and (b) $\int_{x_k}^1 H(0, x)\,dx<0$, since $H(0, x)\leqslant 0$ and by (a) it is not identically zero.

We now prove (a), (b), and (c). (a) follows as in [3, Eqs. (7) and (10)] when we note $\partial F(x^*, x)/\partial x_k$ is a polynomial between the knots. Further $F^{m_k} \neq 0$ by Lemma 1. To prove (b):

Let

$$u_i^\delta(x) = \int_{-\infty}^{\infty} K_\delta(x, y)\, u_i(y)\, dy$$

$$K_\delta(x, y) = \frac{1}{2\sqrt{\pi\delta}} \exp[-(x-y)^2/4\delta].$$

We say $u_i^\delta(x)$ is $u_i(x)$ smoothed. For $\delta > 0$, $u_i^\delta$ is an ETP system. Hence

$$F^\delta(x^*, x) = \frac{\bigcup\begin{pmatrix} u_1^\delta, ..., u_N^\delta, & u_{N+1}^\delta \\ \bar{x}_1, ..., \bar{x}_N, & x \end{pmatrix}}{\bigcup\begin{pmatrix} u_1^\delta, ..., u_N^\delta \\ \bar{x}_1, ..., \bar{x}_N \end{pmatrix}}$$

satisfies the assumptions of [2]. In particular

$$\frac{\partial F^\delta}{\partial x_k}(x^*, x) \begin{cases} \geqslant 0, & x < x_k \\ \leqslant 0, & x > x_k \end{cases} \quad \text{a.e. in } [0, 1].$$

For $y \neq a$, we have $\lim_{\delta \downarrow 0} u_j^\delta(y) = u_j(y)$; $\lim_{\delta \downarrow 0} (d/dy)\, u_j^\delta(y) = (d/dy)\, u_j(y)$ for all $j$ so $\lim_{\delta \downarrow 0} (\partial F^\delta/\partial x_k)(x^*, x) \to (\partial F/\partial x_k)(x^*, x)$ for $x_k \neq a$, $x \neq a$. Thus assertion (b) follows.

To prove (c):

Set $F(x^*, x) = u_{N+1}(x) + \sum_{i=1}^N b_i(x_k)\, u_i(x)$. In $D$ we find we can solve for $b_i(x_k)$ by using the set of equations

$$\sum_{i=1}^N b_i(x_k) \frac{d^j}{dx^j} u_i(x_l) = -\frac{d^j}{dx^j} u_{N+1}(x_l), \qquad j = 0, .., m_l - 1, \quad l = 1, ..., r+1. \tag{26}$$

By differentiating (26) with respect to $x_k$, we obtain the set of equations

$$\sum_{i=1}^N \frac{\partial b_i(x_k)}{\partial x_k} \frac{d^j}{dx^j} u_i(x_l) = \begin{cases} -F^{m_k}, & l = k,\ j = m_k - 1, \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

and can use the solution to describe (in $D$)

$$\frac{\partial F}{\partial x_k}(x^*, x) = \sum_{i=1}^N \frac{\partial b_i(x_k)}{\partial x_k} u_i(x).$$

We claim that at $\varepsilon = 0$, $\partial b_d/\partial x_k = 0$ for $d < \tau + (p - l)$.

This follows because if we call the column on the right side of (27), $-F^{m_k}Y$, the only non-zero term in $Y$ is $Y_{\sigma-1+q}=1$. Further the answer is proportional to $F^{m_k}$. To find the proportionality factor we set $F^{m_k}=-1$. If (at $\varepsilon=0$) we solve for $\partial b_d/\partial x_k$ by Cramer's rule then $Y$ is the $d$th column of the numerator. Subtracting it from the $p-l+1$ column of $L$ ($L_{q,p-l+1}=1$) will result in that column $L$ being identically zero. Thus if we evaluate the numerator, as we did the determinant in the proof of Lemma 5 we find it is zero. Since $\bar{v}_{\tau-1}<\bar{v}_\tau=v_f=a$, (c) follows. ∎

We now discuss the case where there are $q$ zeros at $x_k=a-\varepsilon$, $p$ knots at $v_f=a$. $\bar{x}_{\sigma+i}=a-\varepsilon, i=0, ..., q-1$; $\bar{v}_{\tau+j}=a, j=0, 1, ..., p-1$. (See Fig. 2.) Assume $p+q\geqslant m+1$. Then there exists a $q_2$ such that $p+q_2=m+1$, $q\geqslant q_2$; otherwise we are in $D$, even for $\varepsilon=0$.

LEMMA 8. *Under the above circumstances, there is a* $q_1$ *such that*

(a) $\sigma-1+m+q_1=\tau-1+p$,

(b) $q\geqslant q_1\geqslant 1$,

(c) $p\geqslant q_1$,

(d) $q_1\geqslant(q-q_2)+1$ *or* $q_2\geqslant q_3$ *with* $q_3=(q-q_1)+1$.

*Proof.* $\bar{x}_{\sigma+q}=a_1\geqslant a$; hence $\bar{v}_{\sigma+q+m}>a\Rightarrow\sigma+q+m>\tau-1+p$. On the other hand

$$\bar{v}_\tau=a, \qquad \bar{x}_{\sigma+q-1}=a-\varepsilon\Rightarrow\tau>\sigma-1+q.$$

Combining these two inequalities we find

$$(\sigma-1)+m+(q-q_2)+1\leqslant\tau-1+p\leqslant\sigma-1+m+q.$$

Thus if $q_1$ is defined by (a), (b) and (d) follow.

For (c), note $\bar{v}_{\tau-1}=a_2<a, \bar{x}_\sigma=a-\varepsilon\Rightarrow\bar{v}_{\sigma+m}>a-\varepsilon$ or $\sigma+m>\tau-1$. Thus from (a) $p=[(\sigma-1+m)-\tau+1]+q_1\geqslant q_1$.

Note that at $\varepsilon=0$, this implies $\bar{v}_{g+m}=\bar{x}_g$, $g=\sigma+j, j=0, ..., q_1-1$, which is similar to an overlap, when $x_k=a+\varepsilon$, $v_f=a$. So in this case we also call $q_1$ the overlap at $\varepsilon=0$.

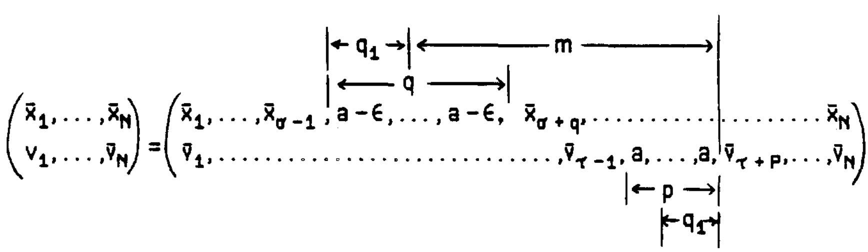


FIG. 2. Overlap $q_1$ at $\varepsilon=0$.

In the present case, we wish to prove, analogously:

LEMMA 11. *Assume in D that $F(x^*, x)$ has a zero of multiplicity $q$ at $x_k = a - \varepsilon$, and that it has a knot of multiplicity $p$ at $v_f = a$, with overlap $q_1$ at $\varepsilon = 0$. Keeping all other zeros fixed, then as a function of $\varepsilon$*

$$\Phi_N = \Phi_m \begin{pmatrix} \bar{x}_1, ..., \bar{x}_N \\ \bar{v}_1, ..., \bar{v}_N \end{pmatrix} = a_z \varepsilon^z + \textit{higher order terms},$$

$$\begin{aligned} a_z \neq 0, \qquad z &= q_1(m - p - q_3 + 1) \\ &= q_1((m + q_1) - (p + q)) \end{aligned}$$

($z \geqslant 0$ *is clear from Fig.* 2).

To prove Lemma 11, we need some preparatory lemmas.

LEMMA 9. *Using the notation of Lemmas* 8 *and* 11, *set $\eta_j = x_k$, in row $\sigma - 1 + j$ of $\Phi_N$, $j = 1, ..., q$, and consider $\Phi_N$ a function of $\eta_1, ..., \eta_q$.*
*Set*

$$D^\alpha \Phi_N = \frac{\partial^{\alpha_1}}{\partial \eta_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_q}}{\partial \eta_q^{\alpha_q}} \Phi_N(\eta_1, ..., \eta_q) \bigg|_{\eta_j = x_k}.$$

*Now replace every term $(x_k - a)_+^0$ that appears in $D^\alpha \Phi_N$ by $z_i$, each such term by a different $z_i$. Let $g$ be the number of such $z_i$'s. Consider $D^\alpha \Phi_N$ as a function of $x_k$ and the $z_i$. Thus $D^\alpha \Phi_N = D^\alpha \Phi_N(x_k, z_1, ..., z_g)$. Finally define*

$$D^\beta D^\alpha \Phi_N(a, b) = \frac{\partial^{\beta_1}}{\partial z_1^{\beta_1}} \cdots \frac{\partial^{\beta_g}}{\partial z_g^{\beta_g}} D^\alpha \Phi_N(a, z_1, ..., z_g) \bigg|_{\text{all } z_i = b}$$

*Then*

(1) $D^\beta D^\alpha \Phi_N(a, 1) = 0$ *if* $|\beta| < q_1$.

(2) $\lim_{x_k \to a-} D^\alpha \Phi_N = D^\alpha \Phi_N(a, 0) = 0$ *if* $g < q_1$.

*Proof.* We consider assertion (2) first. Set $z_i = 1 - \delta$. Then $G(\delta) := D^\alpha \Phi_N(a, z_1, ..., z_g) = D^\alpha \Phi_N(a, 1 - \delta, ..., 1 - \delta)$ is a polynomial of degree $g$ in $\delta$. By Taylor's theorem

$$D^\alpha \Phi_N(a, 0) = G(1) = \sum_{j=0}^{g} \frac{G^j(0)}{j!} = \sum_{j=0}^{g} \frac{(-1)^j}{j!} \sum_{|\beta| = j} \binom{j}{\beta} D^\beta D^\alpha \Phi_N(a, 1).$$

Thus assertion (2) will follow if we prove assertion (1). To prove assertion (1) first note $D^\alpha \Phi_N(a, 1) = 0$.

This follows first from the remark that if you replace $(x_k - v_i)_+^{m_j} = (\text{mat } \Phi_N)_{\sigma, \sigma + q_1 + j}$ by $(x - v_i)^{m_j}$ $(j = 0, ..., m - 1)$, then these $m$ functions span $\pi_m$. Hence if $z_i = 1$, $i = 1 \cdots g$, then subtracting suitable multiples of columns $\sigma + q_1 + j$ ($j = 0, ..., m-1$) from columns $1, .., \sigma + (q_1 - 1)$ results in the latter columns for all rows $\geqslant \sigma$ being zero. Thus the determinant is zero by Laplace's expansion using the first $\sigma$ columns.

If we take one derivative, e.g., $(\partial/\partial z_j)\, D^\alpha \Phi_N(a, z_1, ..., z_g)$ and restrict ourselves to the first $\sigma + (q_1 - 1) - 1$ columns and all rows $\geqslant \sigma$ we can show as above that $(\partial/\partial z_j)\, D^\alpha \Phi_N(a, 1) = 0$. Similarly we can take up to $q_1 - 1$ derivatives to establish our result. (Clearly if any $\beta_i > 1$, $D^\beta \Phi^\alpha(a, b) = 0$, since each $z_i$ appears only to the first power). ∎

Let $M$ be the $q_1 \times q_1$ subdeterminant of $\Phi_N$, formed from rows $\sigma - 1 + i$, $i = q_3, ..., q$, and columns $\tau - 1 + (p - q) + j$, $j = q_3, ..., q$. (See Fig. 3.) Then

$$M = \pm \det \begin{pmatrix} \dfrac{(x_k - a)_+^{m-p-q_3+q+1+(q-q_3)}}{[m-p-q_3+1+(q-q_3)]!} \cdots \dfrac{(x_k - a)_+^{m-p-q_3+1}}{(m-p-q_3+1)!} \\ \dfrac{(x_k - a)_+^{m-p-q_3+1}}{(m-p-q_3+1)!} \cdots \dfrac{(x_k - a)_+^{m-p-q+1}}{(m-p-q+1)!} \end{pmatrix}$$

and $\Phi_{N-q_1}$ is the complementary subdeterminant to $M$ in $\Phi_N$. Then since

$$\Phi_{N-q_1} = \Phi_m \begin{pmatrix} \tilde{x}_1, ..., \tilde{x}_{N-q_1} \\ \tilde{v}_1, ..., \tilde{v}_{N-q_1} \end{pmatrix}$$

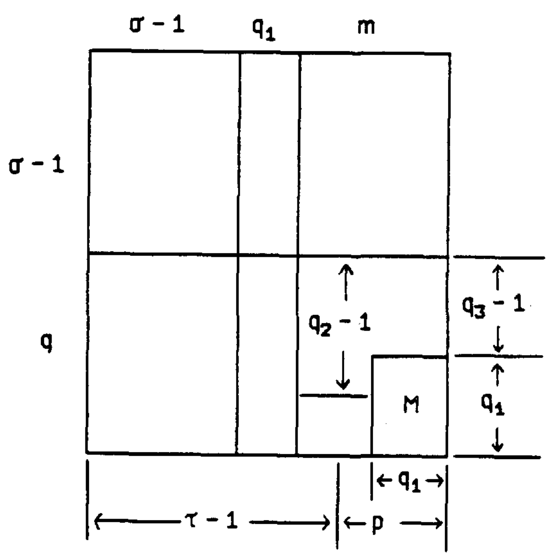


FIG. 3. Upper portion of Mat $\Phi_N$.

with

$$(\tilde{x}_1, ..., \tilde{x}_{N-q_1}) = (\bar{x}_1, ..., \bar{x}_{\sigma+q_3-2}, \bar{x}_{\sigma+q}, ..., \bar{x}_N)$$
$$(\tilde{v}_1, ..., \tilde{v}_{N-q_1}) = (\bar{v}_1, ..., \bar{v}_{(\tau+p)-(q_1+1)}, \bar{v}_{\tau+p}, ..., \bar{v}_N)$$

it is easily verified that

$$\tilde{v}_i < \tilde{x}_i < \tilde{v}_{i+m}.$$

Hence

$$\Phi_{N-q_1} \neq 0. \tag{28}$$

Note that if we set $x_k - a = x$ for $x_k > a$, then $M = W$ where

$$W = \text{Wronskian}\left(\frac{x^{m-p-q_3+1+(q-q_3)}}{[m-p-q_3+1+(q-q_3)]!}, ..., \frac{x^{m-p-q_3+1}}{(m-p-q_3+1)!}\right)$$
$$= x^z \begin{pmatrix} \frac{1}{(m-p-q_3+1+(q-q_3))!}, ..., \frac{1}{(m-p-q_3+1)!} \\ \frac{1}{(m-p-q_3+1)!}, ..., \frac{1}{(m-p-q+1)!} \end{pmatrix} \neq 0,$$
$$z = q_1(m-p-q_3+1), \tag{29}$$

since for $x > 0$ these functions form an ETP system. We will use this fact later.

From Lemma 9, we ask ourselves what is the minimal number of times, call it $\bar{z}$, that we must differentiate $\Phi_N$ so that there are $q_1$ terms of the form $(x_k - a)_+^0$ in different rows and columns. Since $M$ contains the rows and columns of $(x_k - a)_+^j$ to the lowest powers $j$, we see that one way to accomplish this is to differentiate each of the $q_1$ rows occurring in $M$, $(m-p-q_3+1)$ times. Hence $\bar{z} = q_1(m-p-q_3+1) = z$.

LEMMA 10. *Using the notation of Lemmas* 8, 9, *and* 11, *if*

$$\sum_{i=1}^{q} \alpha_i = z = q_1(m-p-q_3+1)$$

*then*

$$\lim_{x_k \to a-} D^\alpha \Phi = (-1)^F D^\alpha M \Phi_{N-q_1} \tag{30}$$

*with both sides zero unless* $\alpha_1 = \alpha_2 = \cdots = \alpha_{q_3-1} = 0$ *and*

$$D^\alpha M = \frac{\partial^{\alpha_{q_3}}}{\partial \eta_{q_3}^{\alpha_{q_3}}} \cdots \frac{\partial^{\alpha_q}}{\partial \eta_q^{\alpha_q}} M(\eta_{q_3}, ..., \eta_q) \Bigg|_{\substack{\text{set all terms } (x_k - a)_+^0 = 1 \\ \text{all other terms} = 0.}}$$

$$F = q_1 + (\sigma + q_3 + \tau + p - q + q_3).$$

*Proof.* Clearly if we differentiate rows or columns not contained in $M$, to get terms of the form $(x_k - a)_+^0$ would require more than $\bar{z}$ derivatives to get $q_1$ such terms. Hence if there are derivatives on the left side of (30) involving rows $\sigma + i$, $i = 0, ..., q_3 - 1$, there will not be $q_1$ terms of the form $(x_k - a)_+^0$ on the left side, so by Lemma 9 they are zero. Also since $\bar{z}$ is minimal $D^\alpha M$ will not contain $q_1$ ones and hence both sides will be zero. Similarly if the left side of (29) has a $(x_k - a)_+^0$ term in columns $\tau + j$, $j = 0, ..., (p - q_1 - 1)$, both sides of (30) will be zero.

Thus we may restrict to the case where all terms $(x_k - a)_+^0$ occur in rows and columns assigned to $M$. Once again if there are not $q_1$ of them in different rows and columns both sides of (30) are zero.

So finally we are left with the case where all terms of the form $(x_k - a)_+^0$ occur in rows and columns of $M$ and $D^\alpha M = \pm 1$, with $\alpha_1 = \cdots = \alpha_{q_3-1} = 0$.

Proceeding as in Lemma 9, set $z_i = 1 - \delta$

$$\lim_{x_k \to a-} D^\alpha \Phi_N = D^\alpha \Phi_N(a, 0) = G(1)$$

$$= \sum_{j=0}^{q_1} \frac{(-1)^j}{j!} \sum_{|\beta| = j} \binom{\alpha}{\beta} D^\beta D^\alpha \Phi_N(a, 1)$$

which by Lemma 9, becomes

$$= (-1)^{q_1} \frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_q} D^\alpha \Phi_N(a, 1, ..., 1)$$

$$= (-1)^F D^\alpha M \Phi_{N-q_1}.$$

The last equality follows since $(\partial/\partial z_1) \cdots (\partial/\partial z_{q_1}) D^\alpha \Phi(a, 1, ..., 1)$ is just the determinant obtained by eliminating the row and column where $z_1$ occurs, the row and column where $z_2$ occurs, etc., i.e., the cofactor of $M$, with proper sign.

Finally we are ready to prove Lemma 11.

*Proof of Lemma* 11. Setting $\eta_j = (a - \varepsilon)$, $j = 1, ..., q$, in rows $\sigma - 1 + j$ of $\Phi_N$ we have

$$\frac{d^h}{d\varepsilon^h} \Phi_N(\varepsilon) = \frac{(-1)^h}{h!} \sum_{|\alpha| = h} \binom{h}{\alpha} D^\alpha \Phi_N.$$

Hence $(d^h/d\varepsilon^h)\,\Phi_N(0)=0$ for $h<z=\bar{z}$ by Lemma 9 and the minimal property of $\bar{z}$ which implies that if $h<\bar{z}$, then the number of distinct $z_i$'s is $<q_1$. Also

$$\begin{aligned}\frac{d^z}{d\varepsilon^z}\Phi(0)&=\frac{(-1)^z}{z!}\sum_{|\alpha|=z}\binom{z}{\alpha}D^\alpha\Phi\\&=\frac{(-1)^z}{z!}\sum_{|\alpha|=z}(-1)^F\binom{z}{\alpha}D^\alpha M\Phi_{N-q_1}\qquad\text{(by Lemma 10)}\\&=(-1)^F\frac{d^zW}{dx^z}\Phi_{N-q}.\end{aligned}$$

Thus the lemma follows from (28) and (29). ∎

For the case $x_k<a$, Lemma 11 is the analogue of Lemma 5 for $x_k>a$. For $x_k<a$, lemmas analogous to Lemmas 6 and 7 for the case $x_k>a$ may now easily be established. Thus Theorem 1′ is proven.

*Proof of Theorem 2′.* Theorem 2′ follows from an improvement theorem, namely $Q(x^*)$ will get smaller if you pull zeros apart in a proper way. We proved a corresponding result for Extended Totally Positive Kernels in [2, Theorem 1].

Let $x^*=(x_1,\ldots,x_r)\in D$ be the zeros of $F(x^*,x)$ with $x_i$ a zero of multiplicity $m_i$, $i=1,\ldots,r$. To avoid some cumbersome notation we will once again restrict ourselves to the situation where the $m_i$ are even and where two zeros come together in $D$, say $x_k$ and $x_{k+1}$, and they both converge to the point $a$. It is possible that $a$ is a knot also. If the sum of the multiplicities of the zeros and of the knot at $a$ is less than $m-2$, our analysis in [2] can be applied to show that $Q(x^*)$ will not assume its minimum when $x_{k+1}=x_k=a$. To treat the general situation when the sum of the multiplicities of the zeros and the knot is less than or equal to $m$, we rely on Lemmas 12, 13, 14, and 15. Once again, the general situation can be treated by obvious extensions of our method.

Assume $x_k=a$ is a zero of multiplicity $q$ and $v_f=a$ is a knot of multiplicity $p$ of $F$ with $p+q\leqslant m$. Let $\bar{x}_{\sigma+i}=x_k$, $i=0,\ldots,q-1$. Then $\bar{x}_{\bar{\sigma}}$, with $\bar{\sigma}=\sigma+q-1$ is the "last" $x_k$ zero.

In Lemmas 12, 13, 14, and 15 we treat the case $p+q=m$. Then $u_{\bar{\sigma}}(x)=\Phi_m^p(x,v_{f-1})$, $p=n_{f-1}$. The proof for $p+q<m$ proceeds in a similar manner.

LEMMA 12. *In the following, assume all $m_i$ even, all $x_j$ fixed except $x_k$, and assume the notation of the paragraph above, $p+q=m$. Define*

$$F_1(x_k, x) = \frac{\bigcup\begin{pmatrix} u_1 \cdots\cdots\cdots \hat{u}_{N+1} \\ \bar{x}_1 \cdots \hat{\bar{x}}_{\bar{\sigma}} \cdots x \end{pmatrix}}{\bigcup\begin{pmatrix} u_1 \cdots \hat{u}_{\bar{\sigma}} \cdots u_N \\ \bar{x}_1 \cdots \hat{\bar{x}}_{\bar{\sigma}} \cdots \bar{x}_N \end{pmatrix}} = \frac{N_1}{D_1}$$

*with $\hat{u}_{N+1}$ signifying that the term $u_{N+1}$ does not appear, etc., and*

$$F_2(x_k, x) = \frac{\partial F_1(x_k, x)}{\partial x_k} = \frac{D_1(\partial N_1/\partial x_k) - N_1(\partial D_1/\partial x_k)}{D_1^2} = \frac{N_2}{D_2}.$$

*Then*

(1a) $F_1 = u_{\bar{\sigma}}(x) + \sum_{\substack{i=1 \\ i \neq \bar{\sigma}}}^{N} h_i(x_k)\, u_i(x)$

(1b) $$\frac{d^j}{dx^j} F_1(x_k, x)\bigg|_{x = x_l} = \begin{cases} 0, & j = 0, ..., m_l - 1, \quad l \neq k \\ 0, & j = 0, ..., m_k - 2, \quad l = k \\ \neq 0, & j = m_k - 1, \quad l = k \end{cases}$$

(2a) $F_2 = \sum_{\substack{i=1 \\ i \neq \bar{\sigma}}}^{N} c_i(x)\, u_i(x)$

(2b) $$\frac{d^j}{dx^j} F_2(x_k, x)\bigg|_{x = x_l} = \begin{cases} 0, & j = 0, ..., m_l - 1, \quad l \neq k \\ 0, & j = 0, ..., m_k - 3, \quad l = k \\ \neq 0, & j = m_k - 2, \quad l = k \end{cases}$$

(2c) $F_2(a, x) \geqslant 0, \qquad \frac{d^{m_k - 2}}{dx^{m_k - 2}} F_2(a, x)\bigg|_{x = a} > 0.$

*Proof.* The powers of $(x_k - a)_+^j/j!$ that appear in $N_1$ and $D_j$ are $j \geqslant 2$. Those that appear in $N_2$ and $D_2$ are $j \geqslant 1$. Furthermore the powers of $(x - a)_+^j/j!$ that appear in $(d^{m_k}/dx^{m_k})\, F_1(x_k, x)$ and $(d^{m_k - 1}/dx^{m_k - 1})$ $F_2(x_k, x)$ are $j \geqslant 2$.

Hence is clear from the definition of $F_1(x_k, x)$ that (1a) is satisfied and so are the first two lines of (1b).

Set $H(x) = \sum_{i=1}^N l_i u_i(x)$, and assume $H(x)$ satisfies:

$$\frac{d^j}{dx^j} H(x)\bigg|_{x = x_l} = 0, \qquad j = 0, ..., m_l - 1, \quad l = 1, ..., k.$$

Then solving for the $l_j$ by Cramer's rule, it would follow since we are in $D$, that all $l_i = 0$. Thus, we establish that line 3 of (1b) is also true.

If we smooth $F_2$, with all $m_i$ even, it follows that $F_2^\delta(x_k, x) \geqslant 0$. (See Lemma 7.) Since $F_2^\delta(x_k, x)$ approaches $F_2(x_k, x)$ uniformly as $\delta \downarrow 0$ it follows that $F_2(x_k, x) \geqslant 0$, and hence the first line of (2c) is established. Statement (2b) and the second line of (2c) follow by the reasoning of Remark 1 of [3].

LEMMA 13. *In D*

(a) *If* $F_3(x) = \sum_{i=1}^{N} d_i u_i(x)$ *satisfies* (1b) *of Lemma* 12, *then*

$$F_3(x) = c_1 F_1(x);$$

(b) *If* $F_4(x) = \sum_{i=1}^{N} l_1 u_i(x)$, $m_k$ *even satisfies* (2b) *of Lemma* 12, *with*

$$\left.\frac{d^{m_k-2}}{dx^{m_k-2}} F_4(x)\right|_{x=x_k} < 0,$$

*then* $F_4(x) = c_2 F_2(x) + c_3 F_1(x)$, $c_2 < 0$.

*Proof.* (a) By the proper choice of $c_1$

$$\left.\frac{d^{m_k-1}}{dx^{m_k-1}} [F_3(x) - c_1 F_1(x)]\right|_{x=x_k} = 0;$$

hence since we are in $D$, the reasoning we used in Lemma 12 shows

$$F_3(x) = c_1 F_1(x).$$

(b) By (2c) of Lemma 12, there is a $c_2 < 0$ such that

$$\frac{d^{m_k-2}}{dx^{m_k-2}} [F_4(x) - c_2 F_2(x)] = 0.$$

Now apply part (a) to $F_4(x) - c_2 F_2(x)$.

LEMMA 14. *Assume all* $m_i$ *even. Keeping all* $x_j$ *fixed, except* $x_k$, *define*

$$Q(x_k) = \int_0^1 F(x_k, x)\, dx \geqslant 0.$$

*Let* $Q(x_k)$ *be minimized at* $x_k = a$. *Then*

$$\int_0^1 F_1(a, x)\, dx = 0.$$

*Proof.* Since $F = G(x) + \sum_{i=1}^{N} b_i(x_k)\, u_i(x)$

$$\frac{\partial F}{\partial x_k} = \sum_{i=1}^{N} \frac{\partial b_i(x_k)}{\partial x_k} u_i(x). \tag{31}$$

We can solve by Cramer's rule, when $x_k \neq a$ (see Lemma 7), to find

$$\sum_{i=1}^{N} \frac{\partial b_i}{\partial x_k} \frac{d^j}{dx^j} u_i(x_l) = \begin{cases} -F^{m_k}, & l=k, \quad j=m_k-1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus by Lemma 13,

$$\frac{\partial F}{\partial x_k}(x_k, x) = C(x_k)\, F_1(x_k, x) \tag{32}$$

with

$$C(x_k) = \frac{F^{m_k} \bigcup \begin{pmatrix} u_1 \cdots \hat{u}_{\bar{\sigma}} \cdots u_N \\ \bar{x}_1 \cdots \hat{x}_{\bar{\sigma}} \cdots \bar{x}_N \end{pmatrix}}{\bigcup \begin{pmatrix} u_1 \cdots u_N \\ \bar{x}_1 \cdots \bar{x}_N \end{pmatrix}}. \tag{33}$$

($C(x_k)$ is chosen to make the coefficient of $u_{\bar{\sigma}}(x)$ on both sides of (32) match.)

Note that $F_1(x_k, x)$ is continuous in $x_k$.

Combining (33) with the statement that $F$ has a zero at $x_k$ of order $m_k$, we obtain from the convention for counting zeros that

$$\begin{aligned} \operatorname{sgn}\,(\lim_{x_k \to a+} C(x_k)) &= \operatorname{sgn}\,(\lim_{x_k \to a+} F^{m_k}) \\ &= \operatorname{sgn}\,(\lim_{x_k \to a-} F^{m_k}) = \operatorname{sgn}\,(\lim_{x_k \to a-} C(x_k)). \end{aligned}$$

Finally, since $Q(a)$ is a minimum

$$\lim_{x_k \to a+} \frac{\partial Q}{\partial x_k} = \lim_{x_k \to a+} C(x_k) \int_0^1 F_1(x_k, x)\, dx \geqslant 0$$

$$\lim_{x_k \to a-} \frac{\partial Q}{\partial x_k} = \lim_{x_k \to a-} C(x_k) \int_0^1 F_1(x_k, x)\, dx \leqslant 0.$$

We conclude $\int_0^1 F_1(a, x)\, dx = 0$. ▌

We now discuss the case of a knot of multiplicity $p$, and a zero $x_k$ of multiplicity $q$, both at point $a$. We break the zero $x_k$ into two zeros of multiplicity $q_1$ at $s_1 = a - \alpha_1 \varepsilon$, and of multiplicity $q_2$ at $s_2 = a + \alpha_2 \varepsilon$, $q_1 + q_2 = q$ (all even). Further we set

$$q_1 \alpha_1 = q_2 \alpha_2, \qquad \alpha_2 > 0, \tag{34}$$

and consider the case $p + q = m$. (If $p + q < m$, the analysis is simpler.)

LEMMA 15. *Use the notation in the paragraph above. Keeping all $x_j$ fixed except $s_1$ and $s_2$, let $F(\varepsilon, x)$ be the corresponding $F$, and let*

$$Q(\varepsilon)=\int_0^1 F(\varepsilon, x)\, dx.$$

*Then*

$$\left.\frac{\partial Q(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}=0 \tag{35}$$

$$\left.\frac{\partial^2 Q(\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0}<0. \tag{36}$$

*Hence $Q(0)$ is not a minimum.*

*Proof.* With

$$\bar{x}_{\sigma+i}=\begin{cases} a-\alpha_1\varepsilon=s_1, & i=0, ..., q_1-1 \\ a+\alpha_2\varepsilon=s_2, & i=q_1, ..., q-1 \end{cases}$$

we write $F$ in the form

$$F=\frac{N_3}{D_3}.$$

By taking divided differences of corresponding terms in both $N_3$ and $D_3$ we can assume the $i$th row of $N_3$ is of the form

$$= \{u_i(x_1), u_i'(x_1), ..., u_i^{m_k-1}(x_{k-1}),$$
$$u_i[1, 0], u_i[2, 0], ..., u_i[q_i, q_2-1], u_i[q_1, q_2], u_i(x_{k+1}), ..., u_i(x)\},$$

where

$$u_i[g, h]=u_i[\overbrace{s_1\cdots s_1}^{g}\ \overbrace{s_2\cdots s_2}^{h}]$$

(the $(g+h-1)$th order divided difference) and similarly for the rows of $D_3$.

From now on we represent $N_3$ in terms of its three most important rows and columns. Thus:

$$N_3=\begin{bmatrix} u_i[q_1, q_2-1] & u_i[q_1, q_2] & u_i(x) \\ (x-a)_+^{m-p+1}[q_1, q_2-1] & (x-a)_+^{m-p+1}[q_1, q_2] & (x-a)_+^{m-p+1} \\ (x-a)_+^{m-p}[q_1, q_2-1] & (x-a)_+^{m-p}[q_1, q_2] & (x-a)_+^{m-p} \end{bmatrix}.$$

Since

$$(x-a)^j_+[1,1]=\frac{(s_2-a)^j_+-(s_1-a)^j_+}{(s_2-s_1)}=\frac{(\alpha_2\varepsilon)^j}{(\alpha_1+\alpha_2)\varepsilon}$$

$$f[x_0, ..., x_m, x, x]=\frac{d}{dx}f[x_0, ..., x_m, x], f[x_0, ..., x_m]=\int_{x_0}^{x_m} B^m(x)\, f^m(x)\, dx,$$

with $B^m(x)$ the standard $B$-spline, it readily follows that

$$N_3=\begin{bmatrix} u_i[q_1, q_2-1] & u_i[q_1, q_2] & u_i(x) \\ A\varepsilon^3 & B\varepsilon^2 & (x-a)^{m-p+1}_+ \\ C\varepsilon^2 & E\varepsilon & (x-a)^{m-p}_+ \end{bmatrix},$$

where $A$, $B$, $C$, $E$ are $>0$. In particular (see Schumaker [11, Eq. 4.39])

$$\begin{aligned} C\varepsilon^2 &= (x-a)^{m-p}_+[q_1, q_2-1] \\ &= \int_{s_1}^{s_2} B^{q-2}(x)\,\frac{d^{q-2}}{dx^{q-2}}(x-a)^{m-p}_+\,dx \\ &= \frac{(m-p)!}{2!(m-p-3)!}\int_{s_1}^{s_2} B^{q-2}(x)(x-a)^2_+\,dx \\ &< \frac{m-p}{2!(m-p-3)!}\int_{s_1}^{s_2} B^{q-2}(x)(x-a)^2\,dx \\ &= (1/2)(q_1\alpha_1^2+q_2\alpha_2^2)\,\varepsilon^2. \end{aligned} \tag{37}$$

Therefore

$$\lim_{\varepsilon\downarrow 0} N_3=\begin{bmatrix} \dfrac{u_i^{q-2}(a)}{(q-2)!} & \dfrac{u_i^{q-1}(a)}{(q-1)!} & u_i(x) \\ 0 & 0 & (x-a)^{m-p+1}_+ \\ 0 & 0 & (x-a)^{m-p}_+ \end{bmatrix}. \tag{38}$$

Since $d/d\varepsilon=-\alpha_1(\partial/\partial s_1)+\alpha_2(\partial/\partial s_2)$, $(d/d\varepsilon)\,u_i[g,h]=-g\alpha_1 u_i[g+1,h]+h\alpha_2 u_i[g,h+1]$; in particular, $\lim_{\varepsilon\downarrow 0}(d/d\varepsilon)\,u_i[q_1,q_2]=(u_i^q(a)/q!)(q_2\alpha_2-q_1\alpha_1)=0$. Hence, we find

$$\lim_{\varepsilon\downarrow 0}\frac{dN_3}{d\varepsilon}=\begin{bmatrix} \dfrac{u_i^{q-2}(a)}{(q-2)!} & 0 & u_i(x) \\ 0 & 0 & (x-a)^{m-p+1}_+ \\ 0 & E & (x-a)^{m-p}_+ \end{bmatrix} \tag{39}$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{d^2 N_3}{d\varepsilon^2} = \begin{bmatrix} \frac{u_i^q(a)}{q!}(q_1\alpha_1^2 + q_2\alpha_2^2) & \frac{u_i^{q-1}(a)}{(q-1)!} & u_i(x) \\ 0 & 0 & (x-a)_+^{m-p+1} \\ 2C & 0 & (x-a)_+^{m-p} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{u^{q-1}(a)}{(q-1)!} & u_i(x) \\ 0 & 0 & (x-a)_+^{m-p+1} \\ 2\alpha_2 E & 0 & (x-a)_+^{m-p} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{u_i^{q-2}(a)}{(q-2)!} & \frac{(q_1\alpha_1^2 + q_2\alpha_2^2)\, u_i^{q+1}(a)}{(q+1)!} & u_i(x) \\ 0 & 2B & (x-a)_+^{m-p+1} \\ 0 & 0 & (x-a)_+^{m-p} \end{bmatrix}.$$

Since the first two determinants only differ in one column, we can "add them" to obtain

$$\lim_{\varepsilon \downarrow 0} \frac{d^2 N_3}{d\varepsilon^2} = (q_1\alpha_1^2 + q_2\alpha_2^2) \begin{bmatrix} \frac{u_i^q(a)}{q!} & \frac{u_i^{q-1}(a)}{(q-1)!} & u_i(x) \\ 0 & 0 & (x-a)_+^{m-p+1} \\ \frac{2C + 2E\alpha_2}{q_1\alpha_1^2 + q_2\alpha_2^2} & 0 & (x-a)_+^{m-p} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{u_i^{q-2}(a)}{(q-2)!} & \frac{u_i^{q+1}(a)}{(q+1)!}(q_1\alpha_1^2 + (q_2\alpha_2^2) & u_i(x) \\ 0 & 2B & (x-a)_+^{m-p+1} \\ 0 & 0 & (x-a)_+^{m-p} \end{bmatrix}$$

$$= A_1 + A_2,$$

where,

$$\text{(a)}\quad \left.\frac{d^j}{dx^j}A_2(x)\right|_{x=a}=0, \qquad j=0\cdots q-2$$

$$\text{(b)}\quad \left.\frac{d^j}{dx^j}A_1(x)\right|_{x=a}=0, \qquad j=0\cdots q-3$$

$$\text{(c)}\quad \left.\frac{d^{q-2}}{dx^{q-2}}A_1(x)\right|_{x=a}$$

$$=(q_1\alpha_1^2+q_2\alpha_2^2)\begin{bmatrix} \dfrac{u_i^q(a)}{q!} & \dfrac{u_i^{q-1}(a)}{(q-1)!} & u_i^{q-2}(a) \\ 0 & 0 & 0 \\ \dfrac{2C+2E\alpha_2}{q_1\alpha_1^2+q_2\alpha_2^2} & 0 & 0 \end{bmatrix}<0. \tag{40}$$

To verify (c) first let $\tilde{F}(x_k, x)=\tilde{N}(x_k, x)/\tilde{D}(x_k)$ be the $F$ with the same zeros and knots as $F(\varepsilon, x)$ except at $x_k$ where $\tilde{F}(x_k, x)$ has $q$ zeros. Next note that the determinant in (40) is obviously an affine function of $K=(2C+2E\alpha_2)/(q_1\alpha_1^2+q_2\alpha_2^2)$. We denote the affine function by $A(K)$.

Since we are in $D$, using the zero count convention we find

$$A(1)=-\lim_{x_k\to a^+}\left.\frac{d^q}{dx^q}\tilde{N}(x_k, x)\right|_{x=x_k}=-\tilde{D}(a)\lim_{x_k\to a^+}\left.\frac{d^q}{dx^q}\tilde{F}(x_k, x)\right|_{x=x_k}<0$$

$$A(0)=-\lim_{x_k\to a^-}\left.\frac{d^q}{dx^q}\tilde{N}(x_k, x)\right|_{x=x_k}=-\tilde{D}(a)\lim_{x_k\to a^-}\left.\frac{d^q}{dx^q}\tilde{F}(x_k, x)\right|_{x=x_k}>0.$$

Thus by (37) $A(2C/(q_1\alpha_1^2+q_2\alpha_2^2))<0$, and hence the determinant may be assumed to be negative, establishing (c). Since

$$\frac{dF}{d\varepsilon}=\frac{D_3(dN_3/d\varepsilon)-N_3(dD_3/d\varepsilon)}{D_3^2}$$

we see by (38) and (39) that $dF/d\varepsilon$ satisfies (1b) of Lemma 12. Hence $dF/d\varepsilon=c_1F_1(a, x)$ by Lemma 13, and thus (35) follows from Lemma 14. Since

$$\frac{d^2F}{d\varepsilon^2}=\frac{D_3(d^2N_3/d\varepsilon^2)-N_3(d^2D_3/d\varepsilon^2)}{D_3^2}$$
$$-\frac{2(dD_3/d\varepsilon)(D_3(dN_3/d\varepsilon)-N_3(dD_3/d\varepsilon))}{D_3^3}$$

it follows by (38), (39), and (40) that $d^2F/d\varepsilon^2$ satisfies part (b) of Lemma 13; hence, $d^2F/d\varepsilon^2 = c_2F_2(x) + c_3F_1(x)$, $c_2 < 0$. Thus by part (2c) of Lemma 12 and Lemma 14, (36) follows. ▮

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